MODULE – VI

PRANDTL'S METHOD

TORSION OF THIN WALLED TUBES

TORSION OF MULTIPLY CONNECTED CELLS

PRANDTL'S METHOD

FOR THE TORSION ANALYSIS OF SOLID SHAFT

TORSION OF NON-CIRCULAR BARS:

Prandtl's Stress Function Method

Prandtl's Stress function ϕ is defined as

$$\tau_{zx} = \frac{\partial \Phi}{\partial y} \qquad \tau_{zy} = -\frac{\partial \Phi}{\partial x} \qquad \textbf{(1)}$$

When, $\boldsymbol{\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0}$; $\tau_{xy} = 0$; τ_{zx} and τ_{zy} are the only non vanishing components of stress the equations of elasticity are given below

Equilibrium Equation

$$\frac{\partial \tau_{zx}}{\partial z} = \frac{\partial \tau_{zy}}{\partial z} = 0$$

(Since the stresses are same in every cross section)

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} = 0$$

(From the third equilibrium equation)

TORSION OF NON-CIRCULAR BARS:

Hooke's Law

The strain components $\boldsymbol{\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \gamma_{xy} = 0}$

$$\gamma_{zy} = \frac{1}{G} \cdot \tau_{zy}$$

$$\gamma_{zx} = \frac{1}{G} \cdot \tau_{zx}$$

Substituting for stress components τ_{zx} and τ_{yz} from equ.1

$$\gamma_{zy} = -\frac{1}{G} \cdot \frac{\partial \Phi}{\partial x}$$

$$\gamma_{zx} = \frac{1}{G} \cdot \frac{\partial \Phi}{\partial y} \qquad \textbf{(2)}$$

TORSION OF NON-CIRCULAR BARS:

Compatibility Equation

The two relevant Compatibility equations for this problem are

$$\frac{\partial}{\partial x}\left(\frac{\partial \gamma_{zx}}{\partial y}+\cancelto{0}{\frac{\partial \gamma_{xy}}{\partial z}}-\frac{\partial \gamma_{yz}}{\partial x}\right)=2\,\cancelto{0}{\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z}}$$

$$\frac{\partial}{\partial y}\left(\cancelto{0}{\frac{\partial \gamma_{xy}}{\partial z}}+\frac{\partial \gamma_{yz}}{\partial x}-\frac{\partial \gamma_{zx}}{\partial y}\right)=2\,\cancelto{0}{\frac{\partial^2 \varepsilon_{yy}}{\partial x \partial z}}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \gamma_{zx}}{\partial y}-\frac{\partial \gamma_{yz}}{\partial x}\right)=0$$

$$\frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x}-\frac{\partial \gamma_{zx}}{\partial y}\right)=0$$

TORSION OF NON-CIRCULAR BARS:

Substituting for γ_{yz} and γ_{xz} from equ. (2) the above two equs becomes

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2}\right)=0$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2}\right)=0$$

$$\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2}=F\ (\text{a constant}) \quad \text{———} \quad \textbf{(3)}$$

TORSION OF NON-CIRCULAR BARS:

Boundary Condition

$$n_x\tau_{xz} + n_y\tau_{yz} + n_z\cancelto{0}{\sigma_{zz}} = \cancelto{0}{F_z}$$

This gives

$$n_x\frac{\partial\Phi}{\partial y} - n_y\frac{\partial\Phi}{\partial x} = 0$$

$$\frac{\partial\Phi}{\partial y}\frac{dy}{ds} + \frac{\partial\Phi}{\partial x}\frac{dx}{ds} = 0$$

$$\frac{d\Phi}{ds} = 0$$

Φ is a constant around the boundary.

TORSION OF NON-CIRCULAR BARS:

For a solid section, $\Phi = 0$ —— **(I)**

around the boundary

On the end faces moment about O should be equal to the applied torque.

$$T = \iint_R \left(\tau_{yz}x - \tau_{xz}y\right)\, dx.dy$$

$$= -\iint_R \left(x\frac{\partial\Phi}{\partial x} + y\frac{\partial\Phi}{\partial y}\right)\, dx.dy$$

$$= -\iint_R \left[\left(\frac{\partial(\Phi x)}{\partial x} + \frac{\partial(\Phi y)}{\partial y}\right) - 2\Phi\right]\, dx.dy$$

$$= -\iint_R \left[\left(\frac{\partial(\Phi x)}{\partial x} + \frac{\partial(\Phi y)}{\partial y}\right)\right]\, dx.dy + \iint_R 2\Phi dxdy$$

TORSION OF NON-CIRCULAR BARS:

Using Gauss theorem,

$$\mathrm{T} = \oint_{S} [\Phi.\mathrm{x}.\mathrm{n_x} + \Phi.\mathrm{y}.\mathrm{n_y}]\mathrm{dxdy} + \iint_{R} 2\Phi\mathrm{dxdy}$$

Since Φ is zero around the boundary the first integral becomes zero and the above equation becomes

$$\mathbf{T} = \iint_{\mathbf{R}} \mathbf{2\Phi dxdy} \text{ ——— } \mathbf{(II)}$$

To find out the constant F in equu 3, taking LHS of equ 3 we get

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = \mathbf{G}\left[\frac{\partial\gamma_{zx}}{\partial y} - \frac{\partial\gamma_{yz}}{\partial x}\right]$$

TORSION OF NON-CIRCULAR BARS:

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = \mathbf{G}\left(\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] - \frac{\partial}{\partial x}\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right]\right)$$

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = \mathbf{G}\left(\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right]\right)$$

$$= \mathbf{G}\,\frac{\partial}{\partial z}\,[-2\omega_z]$$

Where ω_z is the rotation of the element at (x,y) about the z axis.

$\frac{\partial}{\partial z}[\omega_z]$ is the rotation per unit length denoted by θ

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = -2G\theta \text{ ——— } \mathbf{(III)}$$

TORSION OF NON-CIRCULAR BARS:

Summary of Prandtls Method:

Φ is a constant around the boundary.

For a solid section, around the boundary

$$\boldsymbol{\Phi = 0} \quad \text{———} \quad \textbf{(I)}$$

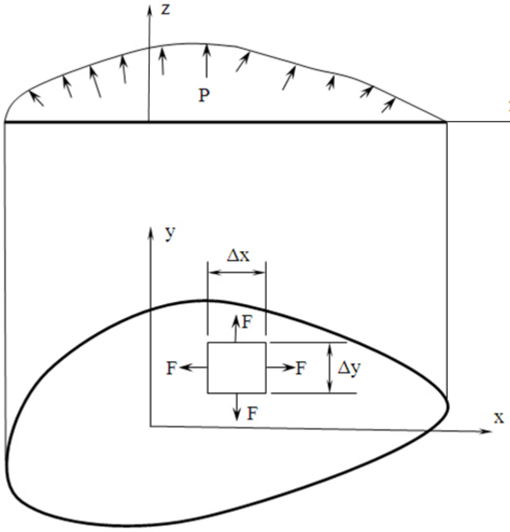
$$\mathbf{T} = \iint_{\mathbf{R}} \mathbf{2\Phi dxdy} \quad \text{———} \quad \textbf{(II)}$$

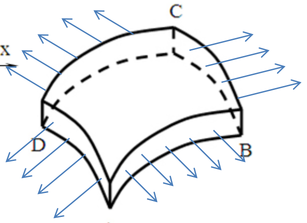
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta \quad \text{———} \quad \textbf{(III)}$$

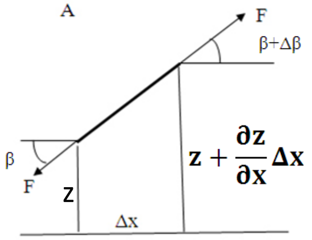
$$\boldsymbol{\tau_{zx}} = \frac{\boldsymbol{\partial\Phi}}{\boldsymbol{\partial y}} \qquad \boldsymbol{\tau_{zy}} = -\frac{\boldsymbol{\partial\Phi}}{\boldsymbol{\partial x}}$$

TORSION OF NON-CIRCULAR BARS:

Prandtl's Membrane Analogy







TORSION OF NON-CIRCULAR BARS:

Prandtl's membrane analogy is very much useful for solving torsion problems involving complicated shapes. Consider a thin homogeneous membrane (like rubber) stressed with uniform tension and fixed at its edges which is having the same shape as the cross section of the shaft in the xy plane.

When the membrane is subjected to uniform lateral pressure P it undergoes a small displacement Z, where Z is a function x & y

Consider the equilibrium of an infinitesimal element ABCD of the membrane after deformation. Let F be the uniform tension per unit length of the membrane.

TORSION OF NON-CIRCULAR BARS:

The value of the initial tension is large enough to ignore its changes when the membrane is blown up by a small pressure.

Force on face AD = F x Δy

Angle made by this force with x axis is β

Slope of AB = $\tan\beta = \frac{\partial z}{\partial x}$

Hence the component of FΔy along z axis $= F\Delta y \frac{\partial z}{\partial x}$

(for very small angle Sin β = tan β)

Force on BC = F x Δy

Angle made by this face with x axis is β + Δ β

TORSION OF NON-CIRCULAR BARS:

Slope at B $= \frac{\partial z}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\Delta x$

The component of the force in the z direction is $F\Delta y\left[\frac{\partial z}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\Delta x\right]$

Similarly the component of force FΔx acting on faces AB and CD are

$F\Delta x\frac{\partial z}{\partial y}$ and $F\Delta x\left[\frac{\partial z}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\Delta y\right]$

The resultant force in the z direction is

$$-F\Delta y\frac{\partial z}{\partial x} + F\Delta y\left[\frac{\partial z}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\Delta x\right] - F\Delta x\frac{\partial z}{\partial y} + F\Delta x\left[\frac{\partial z}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\Delta y\right]$$

$$= F\,\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\Delta x\Delta y$$

TORSION OF NON-CIRCULAR BARS:

The force due to the pressure P acting on the membrane element ABCD is PΔxΔy.

For equilibrium

$$F\left[\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right]\Delta x\Delta y = -P\Delta x\Delta y$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{P}{F}$$

Comparing the above equ. With the Prandtl’s Torsion equ.

$$\frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} = -2G\theta$$

TORSION OF NON-CIRCULAR BARS:

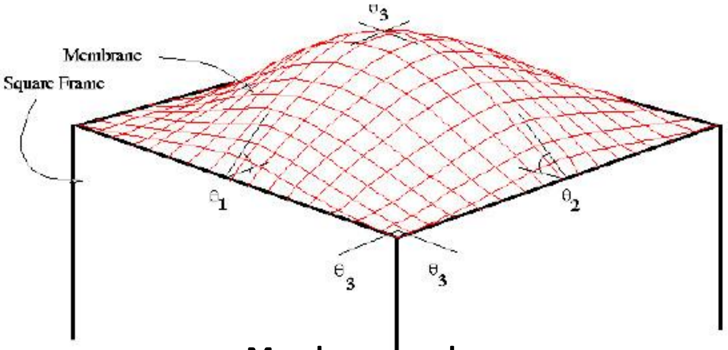
If we adjust the membrane tension F or the air pressure P such that P/F becomes numerically equal to 2Gθ the above two equations becomes identical.

If the membrane height z remains zero at the boundary contour of the section, then the height z of the membrane becomes numerically equal to the torsion stress function.

The slopes of the membrane are then equal to the shear stresses. The twist moment is numerically equal to twice the volume under the membrane.

$$\tau_{zx} = \frac{\partial \Phi}{\partial y} \qquad \tau_{zy} = -\frac{\partial \Phi}{\partial x} \qquad T = \iint_R 2\Phi dxdy$$

TORSION OF NON-CIRCULAR BARS:



Membrane analogy

A steel plate with a square hole was used. Rubber sheet was rigidly clamped at the edges of the hole and made to bulge by applying pressure from beneath the plate. The resulting bulges (torsional hills) for the square hole is shown in Figures.

TORSION OF NON-CIRCULAR BARS:

Solving Torsion of Elliptical Shaft Using Prandtls Method:

The torsion equ $\frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} = -2G\theta$ and boundary condition **Φ = 0** are satisfied by assuming:

$$\emptyset = m\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right]$$

Where m is constant whose value has to be determined.

$$\nabla^2 \emptyset = \frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} = -2G\theta$$

$$\frac{\partial \emptyset}{\partial x} = m\left(\frac{2x}{a^2}\right) \qquad \frac{\partial^2 \emptyset}{\partial x^2} = \frac{2m}{a^2}$$

$$\frac{\partial \emptyset}{\partial y} = m\left(\frac{2y}{b^2}\right) \qquad \frac{\partial^2 \emptyset}{\partial y^2} = \frac{2m}{b^2}$$

TORSION OF NON-CIRCULAR BARS:

Substituting in the torsion equation we get

$$2m\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = -2G\theta \qquad m = -\left(\frac{a^2 b^2}{a^2+b^2}\right)G\theta$$

Substituting for m in the expression for ϕ:

$$\emptyset = -\left(\frac{a^2b^2}{a^2+b^2}\right)\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right]G\theta$$

The shear stress components are:

$$\tau_{zx} = \frac{\partial \emptyset}{\partial y} = -\left(\frac{a^2b^2}{a^2+b^2}\right)\frac{2y}{b^2}G\theta \quad = -\left(\frac{2a^2}{a^2+b^2}\right)G\theta y$$

$$\tau_{zy} = -\frac{\partial \emptyset}{\partial x} = \left(\frac{a^2b^2}{a^2+b^2}\right)\frac{2x}{b^2}G\theta \quad = \left(\frac{2b^2}{a^2+b^2}\right)G\theta x$$

TORSION OF NON-CIRCULAR BARS:

Resultant stress is given by:

$$\tau = \sqrt{\tau_{yz}^2 + \tau_{xz}^2} = \frac{\pm 2G\theta}{\sqrt{a^2 + b^2}}\sqrt{a^4y^2 + b^4x^2}$$

Equating the moment due to shear stress to the external torque:

$$T = \iint_A (\tau_{zy}x - \tau_{zx}y)dA$$

$$T = \iint_A \left(\left(\frac{2b^2}{a^2 + b^2}\right)G\theta x^2 + \left(\frac{2a^2}{a^2+b^2}\right)G\theta y^2\right)dA$$

$$= \frac{2G\theta}{a^2+b^2}\iint_A (a^2y^2 + b^2x^2)dA$$

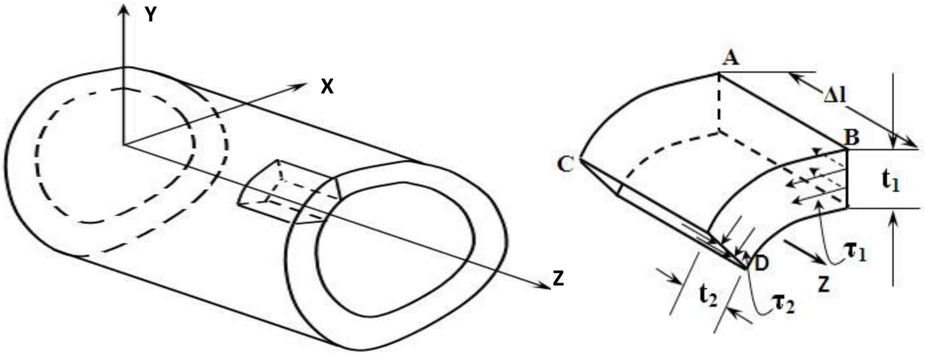
TORSION OF NON-CIRCULAR BARS:

We know that, $I_{yy} = \iint_A x^2 \, dA$ $\quad I_{yy} = \frac{\pi b a^3}{4}$

$$I_{xx} = \iint_A y^2 \, dA \qquad I_{xx} = \frac{\pi a b^3}{4}$$

TORSION OF THIN WALLED TUBES:

Torsion of Thin Walled Tubes



TORSION OF THIN WALLED TUBES:

Force acting on the elementary length ΔS

$$\begin{aligned}\Delta F &= \tau t \Delta S\\ &= q\Delta S\end{aligned}$$

Torque due to the shear stress on the elemental length

$$\begin{aligned}\Delta T &= q\Delta S h.\\ &= 2q\Delta A\end{aligned}$$

ΔA is the area of the triangle enclosed at O by the arc ΔS. Hence the total torque,

$$\begin{aligned}T &= \sum 2q\Delta A\\ &= 2qA\end{aligned}$$

TORSION OF THIN WALLED TUBES:

Strain energy stored due to torsion of elemental length

$$\Delta U = \frac{1}{2}(\tau t \Delta S)\delta$$

$$= \frac{1}{2}(\tau t \Delta S)\ \gamma\ \Delta l$$

$$= \frac{1}{2}(\tau t \Delta S)\ \frac{\tau}{G}\ \Delta l$$

$$= \frac{q^2 \Delta l}{2G}\ \frac{\Delta S}{t}$$

The total strain energy for the tube per unit length

$$U = \frac{T^2}{8A^2 G} \oint \frac{ds}{t}$$ (since T = 2qA)

TORSION OF THIN WALLED TUBES:

Twist per unit length ($\Delta l = 1$)

$$\theta = \frac{dU}{dT}$$

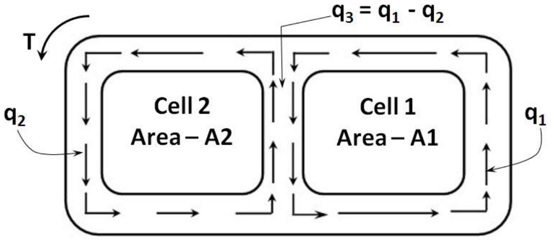
$$\theta = \frac{d}{dT}\left(\frac{T^2}{8A^2 G} \oint \frac{ds}{t}\right)$$

$$\theta = \frac{q}{2AG} \oint \frac{ds}{t}$$ **(since T = 2qA)**

Castiglano's Theorem: If the strain energy 'U' of a linearly elastic structure is expressed as function of generalized force Q_i, then the first partial derivative of U with respect to any one of the generalized force Q_i is equal to the corresponding generalized displacement q_i.

TORSION OF THIN WALLED TUBES:

Torsion of Thin Walled Multiple Cell Closed Sections



Torque for the entire section, $\mathbf{T = T_1 + T_2}$

$$\mathbf{T = 2\,A_1 q_1 + 2A_2 q_2} \qquad \text{———} \quad \mathbf{(1)}$$

(Bredt Batho Equ)

Then $\mathbf{2G\theta = \frac{1}{A} \oint \frac{qds}{t}}$

TORSION OF THIN WALLED TUBES:

Let,

$\mathbf{a_1 = \oint \frac{ds}{t}}$ for cell1 including web

$\mathbf{a_2 = \oint \frac{ds}{t}}$ for cell 2 including web

$\mathbf{a_{12} = \oint \frac{ds}{t}}$ for web

Then for cell 1 and cell2 equ can be written as

$$\mathbf{2G\theta = \frac{1}{A_1}\,[a_1 q_1 - a_{12} q_2]} \qquad \text{———} \quad \mathbf{(2)}$$

$$\mathbf{2G\theta = \frac{1}{A_2}\,[a_2 q_2 - a_{12} q_1]} \qquad \text{———} \quad \mathbf{(3)}$$

Equs 1, 2 &3 can be used to find out q_1, q_2, & q_3

TORSION OF THIN WALLED TUBES:

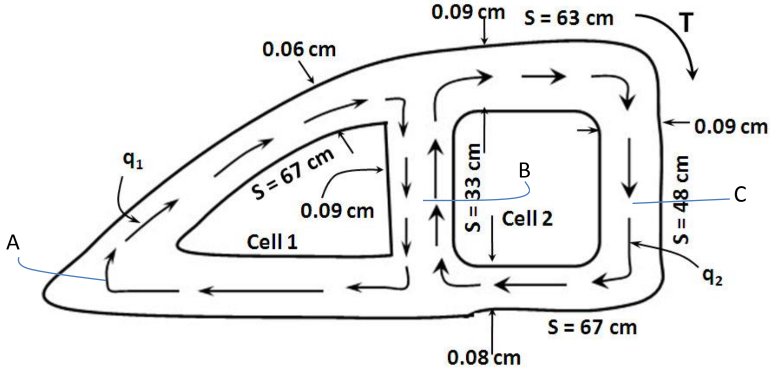
$$T = 2A_1q_1 + 2A_2q_2 \qquad (1)$$

$$2G\theta = \frac{1}{A_1}\left[a_1q_1 - a_{12}q_2\right] \qquad (2)$$

$$2G\theta = \frac{1}{A_2}\left[a_2q_2 - a_{12}q_1\right] \qquad (3)$$

TORSION OF THIN WALLED TUBES:

A two cell thin walled box as shown in fig below is subjected to a torque of 10000N-m Determine the internal shear flow, the stresses at points A, B and C. Also find the angle of twist of per unit length. $A_1 = 680\ \text{cm}^2$; $A_2 = 2000\ \text{cm}^2$ and $G = 80 \times 10^6$ kPa



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$$2G\theta = \frac{1}{A_1}[a_1q_1 + a_{12}q_2] \qquad T = 2A_1q_1 + 2A_2q_2$$

$$2G\theta = \frac{1}{A_2}[a_2q_2 + a_{12}q_1] \qquad q_1 = 14940\text{N/m}$$

$$q_2 = 19920.3\text{N/m}$$

$$q_1 = 0.75q_2$$

$$\tau_A = 24.9\text{ MPa} \qquad \tau_B = 5.534\text{ MPa} \qquad \tau_C = 22.13\text{ MPa}$$

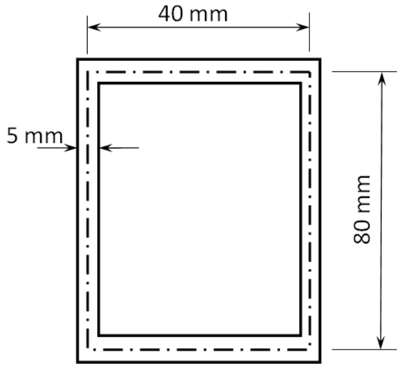
$$\theta = 1.36\text{ radians per m}$$

TORSION OF THIN WALLED TUBES:

Determine the shear stress induced and the angle of twist per unit length of a hollow shaft of dimensions 80 mm x 40 mm and wall thickness 5 mm when subjected to a torque of 1 kN-m. G = 1.3 x 10^4 MPa.

T = 2qA

$$\theta = \frac{q}{2AG}\oint\frac{ds}{t}$$



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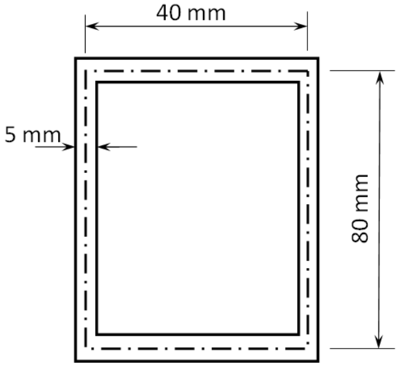
$$A = 3.2 \times 10^{-3}$$

$$q = \frac{T}{2A} = 156250\ \mathrm{N/m}$$

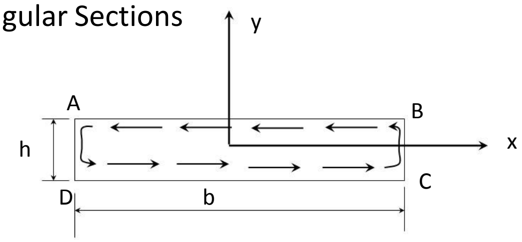
$$\tau = \frac{q}{t} = 31.25\ \mathrm{MPa}$$

$$2G\theta = \frac{q}{A}\oint \frac{dS}{t} = \frac{156250}{3.2 \times 10^{-3}} \times 48$$



TORSION OF THIN WALLED TUBES:

Torsion of Bars with Rectangular Sections



Consider a rectangular bar subjected to a torque T. The thickness t be small compared to the width b. The section consists of only one boundary and the value of the stress function ϕ around this boundary is zero.

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = -2G\theta$$

TORSION OF THIN WALLED TUBES:

The stress function is taken independent of x i.e., ϕ (x,y) = ϕ(y).

$$\frac{\partial^2\Phi}{\partial y^2} = -2G\theta$$

Integrating, $\Phi = -G\theta y^2 + a_1 y + a_2$

Since Φ = 0 at the boundary, Φ = 0 at

$$y = \pm {}^{t}/_{2}$$

Substituting $a_1 = 0 \;\&\; a_2 = \frac{G\theta t^2}{4}$

$$\Phi = G\theta\left[\frac{t^2}{4} - y^2\right]$$

TORSION OF THIN WALLED TUBES:

$$\tau_{zy} = -\frac{\partial\Phi}{\partial x} = 0$$

$$\tau_{zx} = \frac{\partial\Phi}{\partial y} = -2G\theta y$$

These shears are shown in fig.

The maximum shear stress occurs at the surfaces $y = \pm {}^{t}/_{2}$

$$(\tau_{zx})_{max} = \pm G\theta t$$

$$T = 2\iint \Phi\,dxdy$$

$$= 2G\theta\int_{-b/2}^{b/2} dx\int_{-t/2}^{t/2}\left[\frac{t^2}{4} - y^2\right]dy$$

TORSION OF THIN WALLED TUBES:

$$\mathbf{T = \frac{1}{3} bt^3\ G\theta}$$

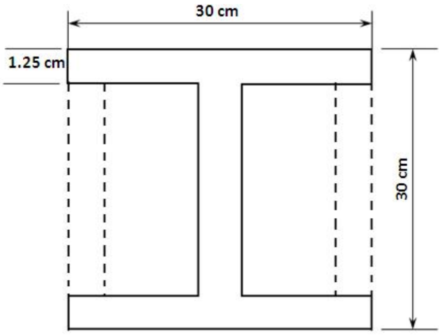
$$\boldsymbol{\tau_{zx} = -2G\theta y}$$

The above equs. Can be applied to rolled sections of different shapes

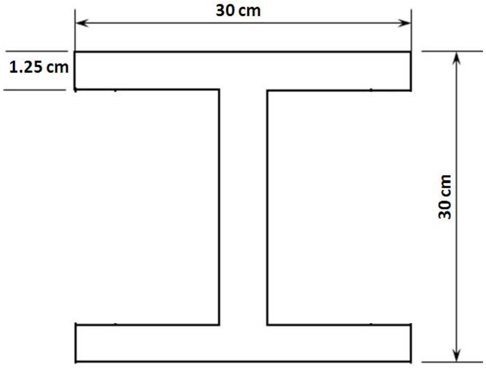
$$\mathbf{T = \sum\left[\frac{1}{3} bt^3\right] G\theta}$$

TORSION OF THIN WALLED TUBES:

A 30 cm I beam with flanges and web 1.5 cm thick s subjected to a torque T = 4900 N-m. Find the maximum shear stress and angle of twist per unit length. In order to reduce the stress and angle of twist, 1.25 cm flat plate are welded on to the sides of the section as shown by dotted lines. Find the maximum shear stress and angle of twist.



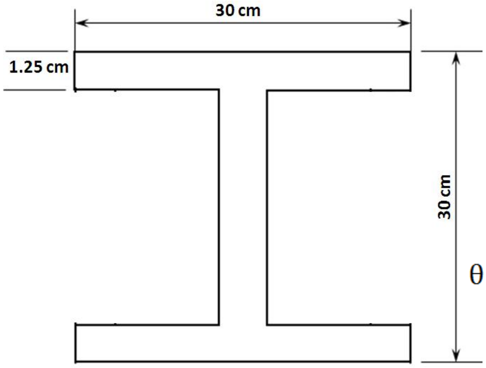
TORSION OF THIN WALLED TUBES:



$$T = \left(\sum \frac{bt^3}{3}\right) G\theta$$

$$\boldsymbol{\tau_{zx} = -2G\theta y}$$

TORSION OF THIN WALLED TUBES:



$$T = \left(\sum \frac{bt^3}{3}\right) G\theta$$

$$\boldsymbol{\tau_{zx} = -2G\theta y}$$

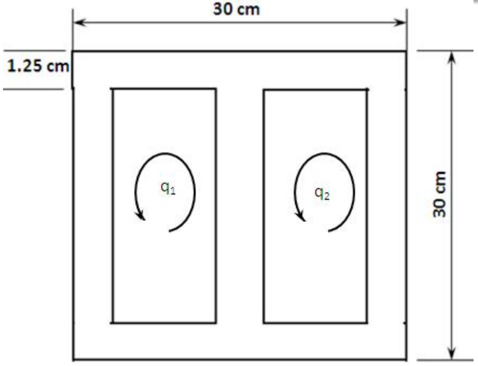
$$\sum \frac{bt^3}{3} = 56.97 \text{ cm}^4$$

$$\theta = 0.1075 \text{ radians per m length}$$

$$\tau_{max} = 107.5 \text{ MPa}$$

TORSION OF THIN WALLED TUBES:

$A_1 = A_2 = 14375 \text{ x } 28.75 = 0.04133 m^2$

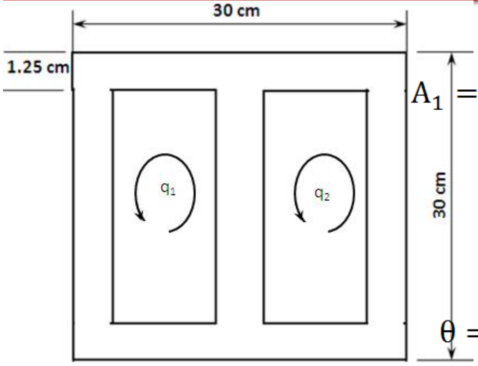


$$T = 2\,A_1 q_1 + 2 A_2 q_2 \qquad \text{——— (1)}$$

$$2G\theta = \frac{1}{A_1}\left[a_1 q_1 - a_{12} q_2\right] \qquad \text{——— (2)}$$

$$2G\theta = \frac{1}{A_2}\left[a_2 q_2 - a_{12} q_1\right] \qquad \text{——— (3)}$$

TORSION OF THIN WALLED TUBES:



$$A_1 = A_2 = 14.375 \text{ x } 28.75 = 0.04133 m^2$$

$$4900 = 4 \text{ x } 0.04133 \text{ x } q$$

$$q = 29641 \text{ N/m}$$

$$\tau_{max} = 2.371 \text{ MPa}$$

$$\theta = 2.063 \text{ x } 10^{-4} \text{ radians per m length}$$

$$T = 2\,A_1 q_1 + 2 A_2 q_2 \qquad \text{——— (1)}$$

$$2G\theta = \frac{1}{A_1}\left[a_1 q_1 - a_{12} q_2\right] \qquad \text{——— (2)}$$

$$2G\theta = \frac{1}{A_2}\left[a_2 q_2 - a_{12} q_1\right] \qquad \text{——— (3)}$$

TORSION OF THIN WALLED TUBES:

Open Section:

$\tau_{max} = 107.512$ MPa

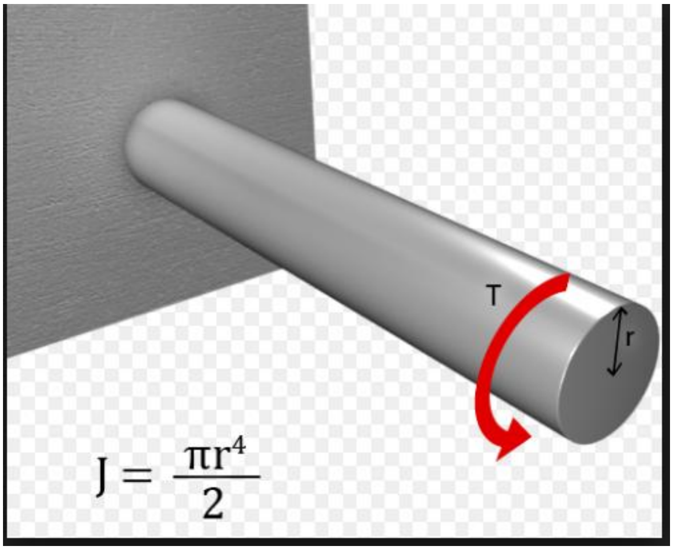
$\theta = 0.1075$ radians per m length

Closed Section:

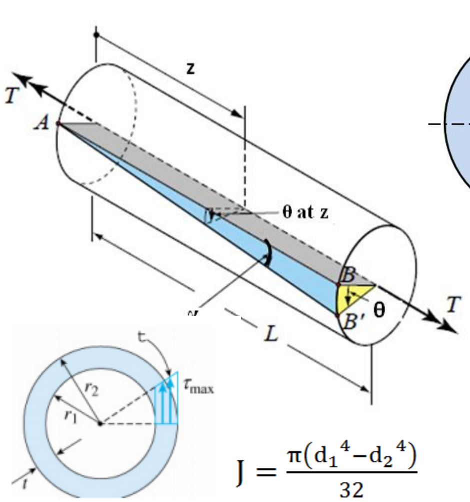
$\tau_{max} = 2.371$MPa

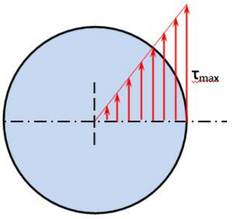
$\theta = 2.063 \times 10^{-4}$ radians per m length

SUMMARY: TORSION OF CIRCULAR SHAFTS



SUMMARY: CIRCULAR SHAFT





$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{l}$$

$$J = \frac{\pi d^4}{32}$$

$$J = \frac{\pi\left({d_1}^4 - {d_2}^4\right)}{32}$$

$$J = \frac{\pi d^3 t}{4}$$

T – Applied Torque in N-m.

J – Polar Moment of Inertia m^4.

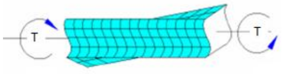
τ – Shear Stress at a radius r in N/m^2.

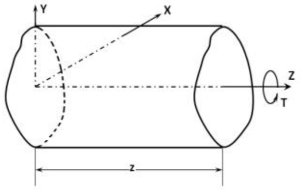
G – Modulus of Rigidity. in N/m^2.

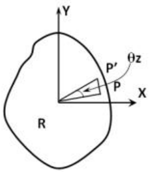
θ – Angular Twist in Radians.

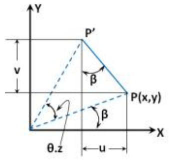
l – length considered in m.

SUMMARY: TORSION OF NON CIRCULAR SHAFT ST VENANT METHOD









$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

$$\left(\frac{\partial\psi}{\partial x} - y\right)\frac{dy}{ds} - \left(\frac{\partial\psi}{\partial y} + x\right)\frac{dx}{ds} = 0$$

$$J = \iint_R \left(x^2 + y^2 + x.\frac{\partial\psi}{\partial y} - y.\frac{\partial\psi}{\partial x}\right) dx.dy$$

$$T = G.J.\theta$$

$$\boldsymbol{\tau_{yz} = G\gamma_{yz} = G\theta}\left(\frac{\partial\psi}{\partial y} + x\right) \qquad \boldsymbol{\tau_{xz} = G\gamma_{xz} = G\theta}\left(\frac{\partial\psi}{\partial x} - y\right)$$

$$u = -\theta.y.z \qquad v = \theta.x.z \qquad w = \theta.\psi(x,y)$$

For Elliptical Section:

$$\tau_{max} = \frac{2T}{\pi ab^2}$$

$$\theta = \frac{T}{G}\frac{a^2 + b^2}{\pi a^3 b^3}$$

SUMMARY: TORSION OF NON CIRCULAR SHAFTS PRANDTLS METHOD

Summary of Prandtls Method:

Φ is a constant around the boundary.

For a solid section, around the boundary

$$\Phi = 0$$

$$T = \iint_R 2\Phi dxdy$$

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 2G\theta$$

$$\tau_{zx} = \frac{\partial \Phi}{\partial y} \qquad \tau_{zy} = -\frac{\partial \Phi}{\partial x}$$

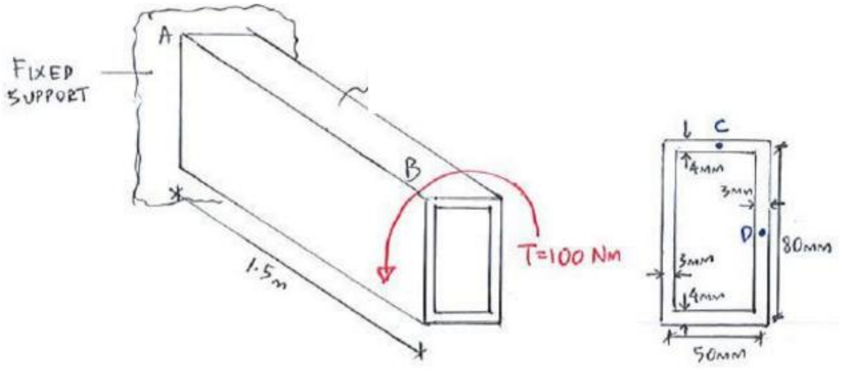
Prandtls Membrane Analogy:

If the membrane height z remains zero at the boundary The twist moment is numerically equal to twice the volume under the membrane.

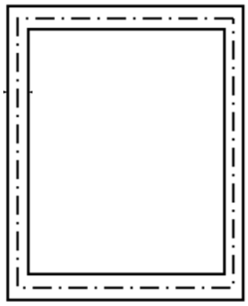
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{P}{F}$$

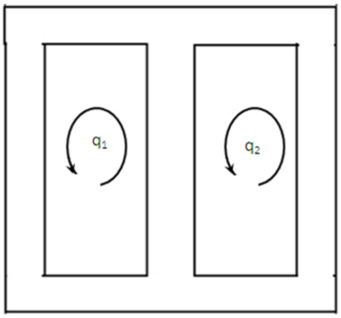
The slopes of the membrane are then equal to the shear stresses.

SUMMARY: THIN WALLED CLOSED SECTION SUBJECTED TO TORSION



SUMMARY: THIN WALLED CLOSED SECTION SUBJECTED TO TORSION





$$T = 2qA$$

$$2G\theta = \frac{q}{A}\oint\frac{dS}{t}$$

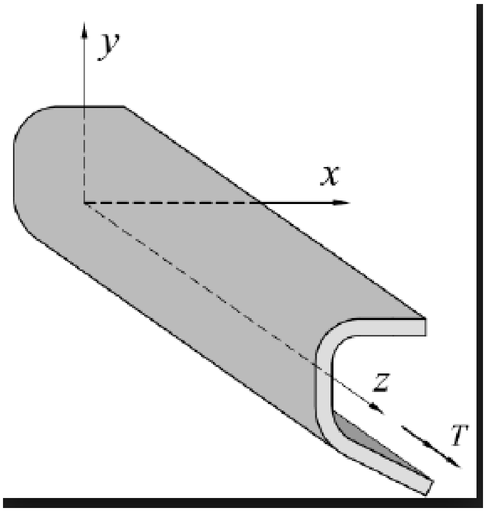
$$T = G.\frac{4A^2}{\oint\frac{ds}{t}}\theta$$

$$T = 2\,A_1q_1 + 2A_2q_2$$

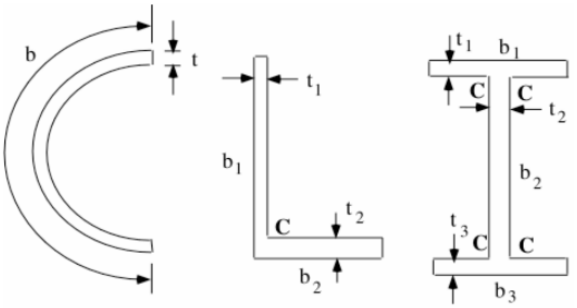
$$2G\theta = \frac{1}{A_1}\left[a_1q_1 - a_{12}q_2\right]$$

$$2G\theta = \frac{1}{A_2}\left[a_2q_2 - a_{12}q_1\right]$$

SUMMARY: THIN WALLED OPEN SECTION SUBJECTED TO TORSION



SUMMARY: THIN WALLED OPEN SECTION SUBJECTED TO TORSION:



$$T = \left(\sum \frac{bt^3}{3}\right) G\theta$$

$$\frac{T}{J} = G\theta$$

$$\boldsymbol{\tau_{zx} = -2G\theta y}$$

$$\tau_{max} = -\frac{Tt_{max}}{J}$$

$$J = \frac{1}{3}bt^3$$

$$J = \frac{1}{3}b_1t_1^3 + \frac{1}{3}b_2t_2^3$$

$$J = \frac{1}{3}b_1t_1^3 + \frac{1}{3}b_2t_2^3 + \frac{1}{3}b_3t_3^3$$

FAQS

1. Discuss the importance of Shear Flow?
2. How will you find torsion effect in a solid bar of non-circular cross-section? Give a standard methodology.
3. Explain Prandtl's method.
4. Explain St. Venant's method.
5. Explain Prandtls. Membrane analogy for torsion of non circular bars.
6. An elliptical shaft of semi axes a =0.05m and b = 0.025 m and G = 800 Gpa is subjected to twsting moment of 1200 π N.m. Determine the maximum shear stress and the angle of twist per unit length
7. Solve the torsion problem of elliptical shaft using Prandtls method.
8. Establish the shape of the prismatic bar for which $\Psi = A(y^3 - 3x^2y)$ is a possible warping function.

FAQS

9. With neat sketches and assumption, derive the torsion formula for any two thin walled open sections.
10. Explain the techniques for finding torsion of thin walled closed section.
11. Derive the expression for torsion of thin walled open and closed sections.
12. Problems discussed in the class